Neural Networks for Learning Counterfactual G-Invariances from Single Environments

a.k.a. "Fixing the Image Rotation Problem"

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Outline

1. Task Description
2. A Lot of Linear Algebra
3. Fun Part

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Introduction

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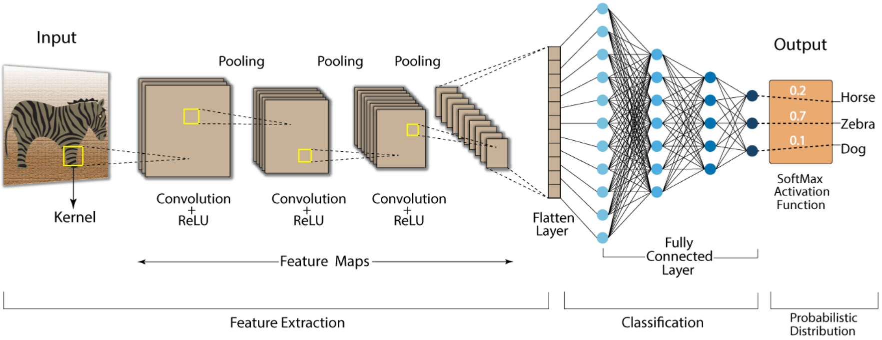

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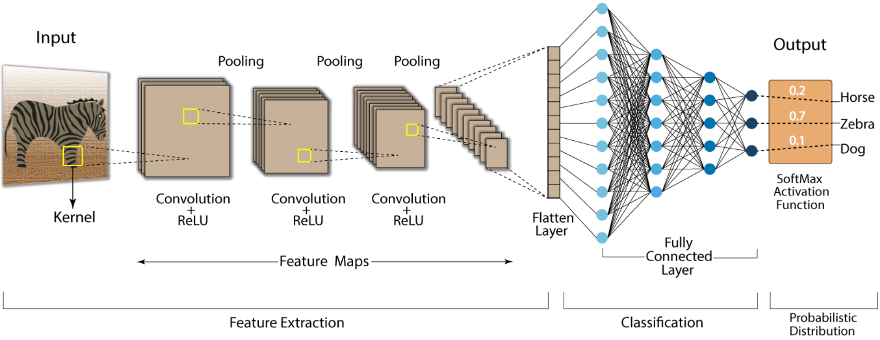

However, they have a <u>critical flaw</u>.

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Today, we'll explore a rotation invariant solution for an MLP. CNN's need **G-equivariance**, which we'll discuss some other time.

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A: These are the axioms on which we define our solution to the rotation problem. Only if these axioms hold true can our solution exist.

The General Linear Group

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Now, we can define *transformations* that can be performed on this group – the only restriction being the axioms of a group.

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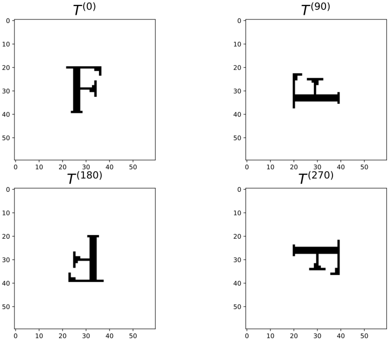

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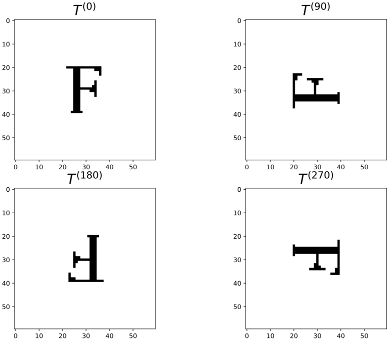

Both groups are defined $G : \mathbb{R}^{3n^2} \to \mathbb{R}^{3n^2}$

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Our objective is to learn the optimal weight on a layer such that

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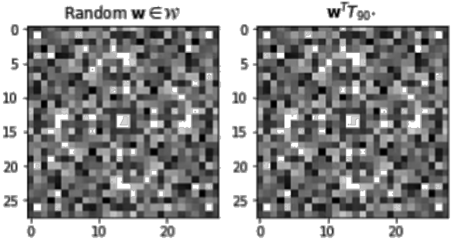

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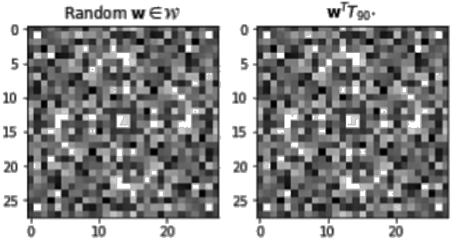

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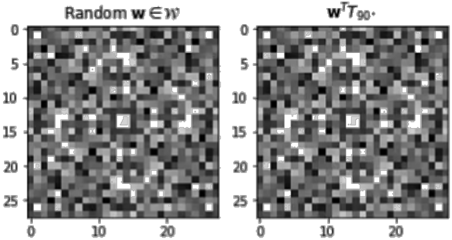

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Q: How can we go about finding this?

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A given transformation is G-invariant if,

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Extracting the left eigenvectors of T_2, we obtain v_n. By definition of the eigenvectors:

$$\begin{aligned} v_i T_2 &= \lambda_i v_i \\ v_i T_2 &= v_i \end{aligned}$$

T_2 is a projection operator

Defining the Output Latent Space

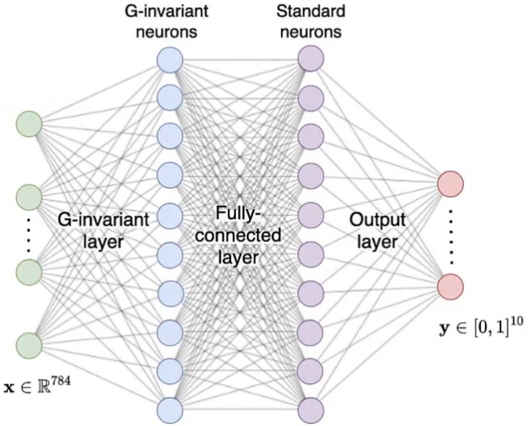
We can now define our weights such that,

$$w^T = \sum_{i=1}^{k} \alpha_i v_i^T$$

where T stands for transpose, & α_i is arbitrary

Bringing it all Together

In order to set up our complete neural network, all we have to do is prepend the G-invariant layer we just built to our standard model.



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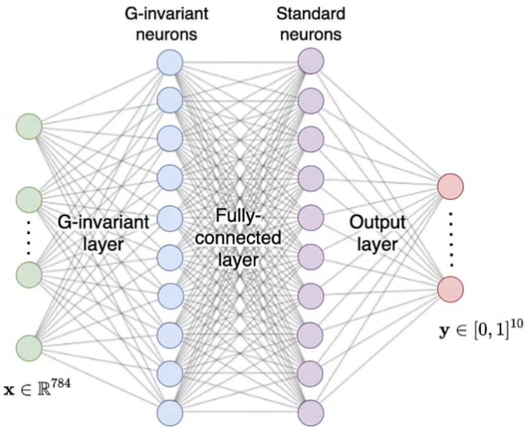

This will ensure that we build a feature space that's invariant to rotation!

Executing Code

Let's classify!!

If you can view this screen, I am making a mistake.

Thank you!

Have an awesome rest of your day!

Slides: `https://cs.purdue.edu/homes/jsetpal/g-invariance.pdf`